Ambiguity Functions of Matched Illumination Radar Signals

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Abstract— Ambiguity function expressions are derived for radar using matched illumination (MI) transmit signals for the detection of range spread targets in the presence of clutter. The transmit signals are adapted to target and interference spectra and are filtered optimally in the receiver; they are designed to maximize received signal to total interference (SINR) power ratios. It is shown here that in the case of extended targets, ambiguity functions resulting from using optimal MI constant envelope waveforms obtained via phase retrieval techniques have superior resolution characteristics compared to linear FM signals employing corresponding optimal pulse compression. Together with the well known fact that optimally filtered adaptive MI signals provide significantly enhanced SINR behavior, this result then sets the stage for the induction of MI signaling and receive techniques into conventional radar signal processing and paves the way for realization of one methodology to achieve cognition in radar systems.

Keywords—Ambiuity Function, Range Spread Target, Matched Illumination.

I. INTRODUCTION

The study of transmit radar waveforms that are chosen to operate optimally in the presence of targets and interference (clutter and additive noise) that have specified or estimated spectral properties has been of interest for several decades, [1]-[6]. These waveforms are designed to typically maximize SINR ratios [4] or the mutual information between received signals and the stochastic target response [3], depending on the radar application under consideration. From the point of view of ambiguity resolution and robustness to interfering targets, it is of great interest to evaluate the ambiguity functions (AF) that result from the use of these MI waveforms and their corresponding optimum receive filters. This paper presents the derivation of the AF of MI signals in the presence of extended targets and clutter. To the best of our knowledge, the derivation of ambiguity function for extended target in presence of clutter hasn't been attempted in literature earlier. It turns out that these spread AFs are functions of target and clutter spectra with the receive processing using optimal filtering that is not the conventional matched filter but again depends upon these spectral functions. In fact, the optimal filter consists of a whitening operation followed by a matched filter, a result that is well known. Simulation results using optimal MI signals (or their constant envelope versions) reveal that, for a large class of extended target spectra, the spread AFs possesses superior performance to the classic linear frequency modulated signals.

II. RADAR SIGNAL MODEL

The MI radar system model used here is the usual one found in the literature, [4]. The extended (range or delay spread) target is represented by a baseband impulse response $h(t)$. For simplicity in deriving expressions for the AF, the target is assumed to be nonfluctuating. The clutter (impulse) response $c(t)$ at baseband is modelled as a complex Gaussian random process with zero mean possessing a covariance $K_c(t-u)$ and corresponding power spectral density $\Phi_c(f)$. The receiver noise process $n(t)$ is complex Gaussian with zero mean and has covariance function $\Phi_n(t-u)$ with power spectral density $\Phi_n(f)$. Both these processes are assumed to be covariance stationary.

The following is a straightforward application of the point target model described in [11] (Chapters 9 and 10), to an extended target with response $h(t)$ and Fourier transform $H(f)$. The transmitted radar signal $f(t)$ consisting of a complex baseband signal $x(t)$ modulating a carrier frequency f_c is

$$f(t) = R\{x(t)e^{j2\pi f_c t}\}, \quad -\infty < t < \infty \tag{1}$$

The signal emanating from the target is a convolution of the target response and the impinging transmitted signal. This reflection is further delayed and is finally received at the radar as

$$y_s(t) = R\{\int h(\lambda)x(t-\tau_a-\lambda)d\lambda . e^{j2\pi(f_c+f_a)t-j2\pi f_c\tau_a}\} \tag{2}$$

subject to the standard narrowband assumption, [10],[11]. The Doppler shift in received carrier frequency has been denoted as $f_a = \left(\frac{2v}{c}\right)f_c$ where v is target velocity, and τ_a is the round-trip delay. Define the target reflection

$$p(t) \equiv k\int h(\lambda)x(t-\lambda)d\lambda \tag{3}$$

where $k = e^{-j2\pi f_c\tau_a}$. The complex envelope $y_{s,r}(t;\tau_a,f_a)$ of the baseband version of the received signal is then

$$y_{s,r}(t;\tau_a,f_a) = p(t-\tau_a)e^{j2\pi f_a t} \tag{4}$$

with frequency spectrum

$$\begin{aligned} Y_{s,r}(f;\tau_a,f_a) &\triangleq F\{y_{s,r}(t;\tau_a,f_a)\} \\ &= P(f-f_a)e^{-j2\pi(f-f_a)\tau_a} \end{aligned} \tag{5}$$

at baseband where $P(f) = kH(f)X(f)$ and $X(f)$ is the Fourier transform of the transmit signal $x(t)$.

Assuming zero relative motion between background clutter and radar, the received clutter process $c_r(t)$ is given by the convolution

$$c_r(t) = k\int_{-\infty}^{\infty} c(\lambda)x(t-\tau_a-\lambda)d\lambda \tag{6}$$

in analogy with (3) and (4). The received clutter power spectral density is thus $|X(f)|^2\Phi_c(f)$ with associated covariance $K_{c,r}(t-u)$. The total received signal $y(t)$ is then

$$\begin{aligned} y(t) &= y_{s,r}(t;\tau_a,f_a) + c_r(t) + n(t) \\ &= p(t-\tau_a)e^{j2\pi f_a t} + c_r(t) + n(t) \end{aligned} \tag{7}$$

Hence the power spectral density $\Phi(f)$ of the total received interference is given by

$$\Phi(f) = |X(f)|^2\Phi_c(f) + \Phi_n(f) \tag{8}$$

A. Optimum receive filter

For the received signal spectrum in (5) and interference power spectrum in (8), the optimum receive filter that maximizes the output signal to interference power ratio SINR at time $t = T_0$ is the standard coloured-noise matched filter (section 5.5,[8]) given by

$$\begin{aligned} R(f) &= \frac{Y^*_{s,r}(f;\tau_a,f_a)}{\Phi(f)} e^{-j2\pi f T_0} \\ &= \frac{P^*(f-f_a)}{\Phi(f)} e^{-j2\pi f (T_0-\tau_a)} \\ &= \frac{H^*(f-f_a)X^*(f-f_a)}{\Phi(f)} e^{-j2\pi f (T_0-\tau_a)} \end{aligned} \tag{9}$$

dropping the constant phase factor $k^* e^{-j2\pi f(T_0-\tau_a)}$ without affecting SINR. The filter is, in general, not causal and assumes that τ_a and f_a are known exactly in the receiver. The signal to interference power ratio at $t = T_0$ is given by

$$SINR(t = T_0) = \int \frac{|P(f-f_a)|^2}{\Phi(f)} df \tag{10}$$

III. Matched Illumination Signals

In matched illumination transmit waveform design, the signal $x(t)$ is chosen to maximize

$$SINR = \int \frac{|P(f)|^2}{\Phi(f)} df = \int \frac{|H(f)|^2|X(f)|^2}{\Phi(f)} df \tag{11}$$

where the Doppler shift f_a is set to zero in (10) to avoid the complication of attempting to make the transmit signal shape adapt to it. It is assumed that knowledge of target response $H(f)$ and interference spectrum $\Phi(f)$ is available. The answer is well known, [3]-[7], and prescribes a water-filling type solution given by

$$|X(f)|^2 = \max\,[0, B(f)(\mu - D(f))] \tag{12}$$

where $B(f)$ and $D(f)$ are

$$B(f) = \frac{|H(f)|\sqrt{\Phi_n(f)}}{\Phi_c(f)} \tag{13}$$

and

$$D(f) = \frac{\sqrt{\Phi_n(f)}}{|H(f)|} \tag{14}$$

The parameter μ determines energy E_t of the transmit waveform as

$$E_t = \int |X(f)|^2 df \tag{15}$$

where the integration is over the bandwidth of the transmit signal $X(f)$. Define the set Ω as

$$\Omega = \{f : \mu > D(f)\}$$

Then

$$|X(f)|^2 = \begin{cases} 0 & f \in \Omega^c \\ B(f)\big(\mu - D(f)\big) & f \in \Omega \end{cases} \tag{16}$$

The corresponding optimum receive filter $R(f)$ is given by (9) with τ_a and f_a set to zero. In the rest of the paper we restrict consideration to white Gaussian noise with $\Phi_n(f) = N_0$. Then, using (8) and (16), the received interference power spectrum becomes

$$\Phi(f) = \begin{cases} N_0 & f \in \Omega^c \\ \mu\sqrt{N_0\,|H(f)|^2} & f \in \Omega \end{cases} \tag{17}$$

by employing the clutter- and target-matched transmit signal $x(t)$.

A. Point Target

For a point target, $|H(f)| = 1$ say. In this case the set Ω is the entire frequency axis, if solution exists, and using (13) and (14) in (16), the matched illumination waveform is

$$|X(f)|^2 = \frac{constant}{\Phi_c(f)} \tag{18}$$

which means that for a point target the transmit waveform attempts to whiten the received clutter spectrum. This is a well known early result, [1]. The optimum receive filter $R(f)$ is then matched to the waveform $X(f)$. □

Detailed analyses of matched illumination signals, their SINR performances, and consideration of various target and clutter models can be found in the comprehensive and fine treatment of the subject contained in [5].

Remarks: Strictly speaking, the (magnitude squared) signal spectrum of (16) alongwith the receive filter (9) for τ_a=0 and $f_a = 0$ maximizes and achieves the SINR in (11) and *not* that in (10). In any implementation of matched illumination processing, it is proposed to use the correlator operation in the first term of the log-likelihood function in (22) as a sufficient statistic, or the equivalent matched filter. Under perfectly matched conditions then, the resulting output SINR achieved using the (translated) transmit signal spectrum of (16) will be that of (10), as pointed out following (25).

IV. LIKELIHOOD FUNCTION

The covariance function $K(t-u)$ of the interference process $c_r(t)+n(t)$ is

$$K(t-u) = F^{-1}\{\Phi(f)\}$$
$$= K_{c,r}(t-u) + N_0\delta(t-u) \tag{19}$$

Following the development of section 4.3 of [9] and section 9.3 of [11], an inverse kernel Q based on the covariance $K(t-u)$ is

$$\int K(s-u)Q(v-s)ds = \delta(u-v) \tag{20}$$

and in frequency domain is

$$F\{Q(\tau)\} \triangleq \Phi_Q(f) = \frac{1}{\Phi(f)} \tag{21}$$

Then the log-likelihood function $L(y(t);\tau_a,f_a)$ of the received signal $y(t)$ in (7) is, within some multiplicative constants, given by

$$L(y(t);\tau_a,f_a) = \int y(t)g^*(t;\tau_a,f_a)dt$$
$$-\frac{1}{2}\int y_{s,r}(t;\tau_a,f_a)g^*(t;\tau_a,f_a)dt \tag{22}$$

This is the complex version of the log-likelihood function in section 4.3.5 of [9]. The correlator signal $g(t)$ is obtained from

$$g(t;\tau_a,f_a) = \int Q(t-u)y_{s,r}(u;\tau_a,f_a)du \tag{23}$$

and transforming

$$G(f;\tau_a,f_a) = F\{g(t;\tau_a,f_a)\}$$
$$= \Phi_Q(f)Y_{s,r}(f;\tau_a,f_a) = \frac{P(f-f_a)}{\Phi(f)}e^{-j2\pi(f-f_a)\tau_a} \tag{24}$$

using (5) and (21). It is easy to see that the correlator output (first term on the right in (22)) is identical to the output of the matched filter (9) at time T_0. A suitably thresholded correlator leads to the optimum detector, while for slowly fluctuating targets the correlator is followed by a square-law device,[10].

V. SPREAD AMBIGUITY FUNCTION

For range spread targets the resulting ambiguity function, denoted here as A_{spread}, will be considered as magnitude (or magnitude squared for fluctuating targets) of the expectation of the correlator (or receive filter) output, in the absence of additive interference. This definition is in line with the description of a *spread ambiguity function* discussed in chapter 11 of [11]. The receiver processing is assumed to be mismatched to the target return in range and Doppler parameters. With this definition therefore, the first term of (22) which is a sufficient statistic for the detection problem, leads to

$$A_{spread}(\tau_h,f_h;\tau_a,f_a) = \left|\int y_{s,r}(t;\tau_a,f_a)g^*(t;\tau_h,f_h)dt\right|$$
$$= \left|\int Y_{s,r}(f;\tau_a,f_a)G^*(f;\tau_h,f_h)df\right|$$
$$= \left|\int \frac{Y_{s,r}(f;\tau_a,f_a)Y^*_{s,r}(f;\tau_h,f_h)}{\Phi(f)}df\right|$$
$$= \left|\int \frac{P(f-f_a)P^*(f-f_h)}{\Phi(f)}e^{-j2\pi f(\tau_a-\tau_h)}df\right| \tag{25}$$

for the spread ambiguity function, where (τ_h,f_h) are the hypothesized parameters in the receiver. The signal spectrum $P(f)$ is a function of target and transmit signal spectra through (3) and the latter depends, through matched illumination, on the clutter spectrum. When $(\tau_h,f_h)=(\tau_a,f_a)$, (25) becomes the SINR expression in (10) and the ambiguity surface reaches its summit. Therefore, choosing transmit signals $x(t)$ that maximize SINR can lead to peaky ambiguity functions.

A. Point Target

For a point target, $p(t = kx(t))$ from (3) and the interference spectrum in (17) becomes white ($|H|=1$). The ambiguity function in (25) reduces, within a constant, to

$$A_{point}(\tau_h,f_h;\tau_a,f_a)$$
$$= \left|\int X(f-f_a)X^*(f-f_h)e^{-j2\pi f(\tau_a-\tau_h)}df\right| \tag{26}$$

where $\tau_m=\tau_h-\tau_a$ and $f_m=f_h-f_a$ denote receiver mismatches in range and Doppler. These are exactly the frequency and time domain versions of the standard Woodward ambiguity function (equation 117, page 309, [11]). The ambiguity function in this case is determined by the transmit signal shape and therefore, through (18), solely by the clutter power spectrum Φ_c.

If matched illumination were not used in the point target case above, then the $\Phi(f)$ term in the denominator of the integrand in (25) would not reduce to a constant with the result that the elegant dependence of the ambiguity function A_{point} solely on the mismatch quantities τ_m and f_m would be lost. This same inference is true for A_{spread} in the general case of a range spread target *with* matched illumination in use. However the following symmetry holds for (25) as can be easily seen

$$A_{spread}(\tau_h,f_h;\tau_a,f_a) = A_{spread}(\tau_a,f_a;\tau_h,f_h) \tag{27}$$

VI. Results & Discussion

We here present the Ambiguity function simulation results for $X(f) = e^{-\pi^2 f^2 T_s^2}$ for $T_s = 20\mu s$, two point corner reflector target spectrum, denoted as a $H(f) = \frac{2a_1b_1}{b_1^2+f^2} + \frac{2a_2b_2}{b_2^2+f^2} e^{j2\pi f\tau}$ with $a_1 = a_2 = 1$, $b_1 = 7.2Mhz, b_2 = 3.7Mhz$ and $\tau = 0.25us$ is used. The clutter power spectrum density $\Phi_c(f) = \frac{k\sigma_c^2}{\pi f^2+\pi k^2}$ with $k = 0.25MHz$ and $\sigma_c^2 = 1.18$ is used. We numerically compute the following ambiguity functions:

- Woodward AF using matched filter (MF) with transmitted $X(f)$ (assuming point target)
- AF resulting from use of same matched filter with transmitted X(f), for spread target H(f) (conventional radar case)
- Spread AF using colored-noise matched filter (CNMF) $G(f;\tau_a, f_a)$ with transmitted $X(f)$ for target H(f)
- Spread AF for CNMF with $G(f;\tau_a, f_a)$ with transmitted SINR-MI $X_{MI}(f)$ for target H(f).

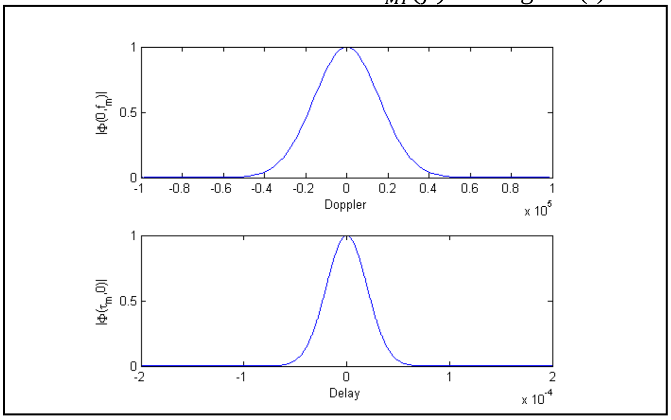


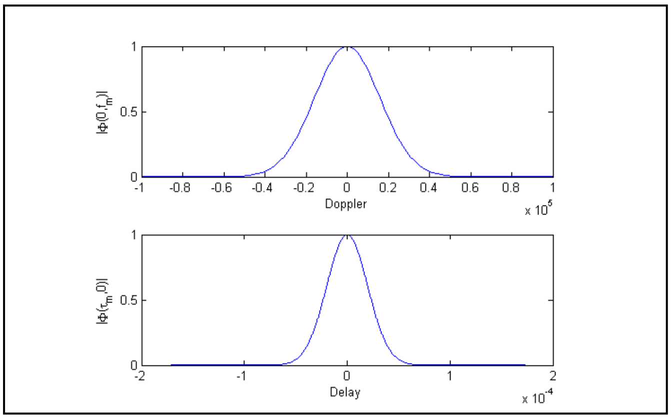
Figure 1 Woodward Ambiguity Function for $X(f)$ top) across $\tau_m = 0$ bottom) across $f_m = 0$

The SINR-MI waveforms are derived based on known target spectrum, clutter PSD, and noise PSD. Noise is assumed to be additive white Gaussian. The clutter-to-noise ratio (CNR) are set at 0 dB. From the prescribed SINR-MI Fourier transform magnitudes $|X(f)|^2$, we use the phase retrieval algorithm proposed in our earlier work [12] to arrive at constant envelope transmit waveforms. The spread - ambiguity functions are computed using these phase retrieved SINR-optimal waveforms.

Table 1 Ambiguity Function Parameters from different cases

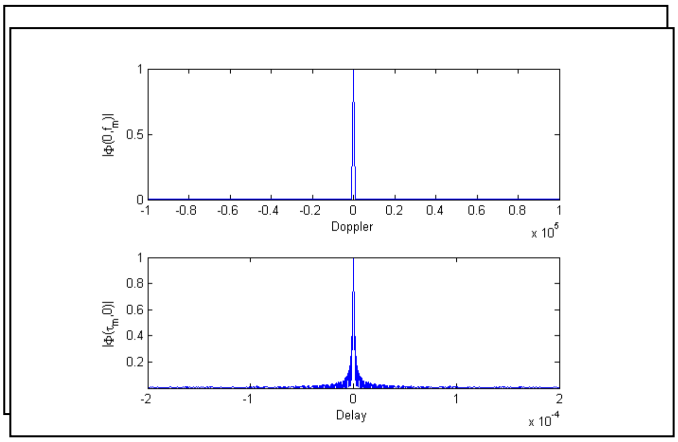
Parameter	Wood-ward	MF- X(f)	CNMF-$X(f)$	CNMF-SINR-MI
3dB Beam-width	33us, 26Khz	33us, 26Khz	27us, 32Khz	<0.5us, <1Khz
First Zero Cross-Overs	95.5us, 76khz	95.5us, 76Khz	153us, 78Khz	110us, 66Khz
Peak Side-lobes	Not seen	Not seen	Not seen	-6.3057, -61.317

Figure 2 Spread Ambiguity Function Slice for MF, $X^*(f;\tau_h, f_h)$



for $X(f)$ top) across $\tau_m = 0$ bottom) across $f_m = 0$

Figure 3 Spread Ambiguity Function Slice for CNMF,



$G(f;\tau_h, f_h)$ for $X(f)$ top) across $\tau_m = 0$ bottom) across $f_m = 0$

Figure 4 Spread Ambiguity Function Slice for CNMF $G(f;\tau_h, f_h)$ for SINR-MI top) across $\tau_m = 0$ bottom) across $f_m = 0$

The phase retrieval algorithm runs for a pre-specified number of iterations (typically 5000) and the spread-ambiguity function is computed numerically. From

Figure 2, Figure 3, **Error! Reference source not found.**, we observe that on using constant envelope SINR-MI waveform the range and Doppler resolution is highly

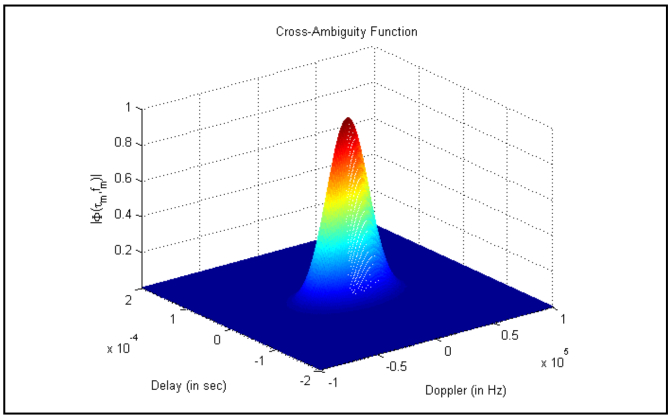


Figure 5 3D Ambiguity Function for Matched Filter

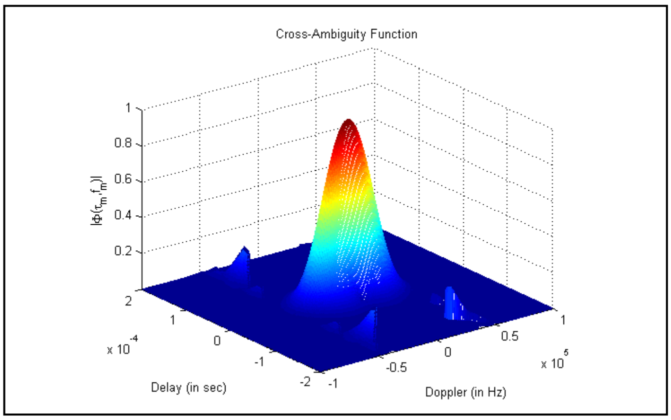


Figure 6 3D Spread Ambiguity Function for Colored Noise Matched Filter for $X(f)$

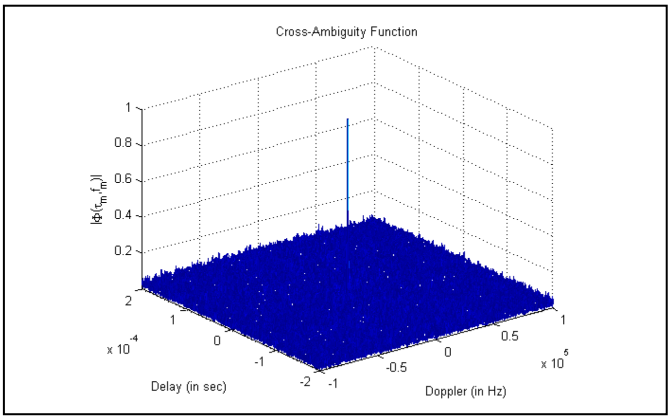


Figure 7 Spread Ambiguity Function for Colored Noise Matched Filter for SINR-MI waveform

improved. The result is also intuitively satisfying since on using SINR-MI waveform at the filter output we have a ~3-4dB higher peak [12][7] and given the fact that ambiguity function volume is preserved or what is known as radar uncertainty principle [11], its 3-dB beamwidth of the ambiguity surface is expected to be proportionally smaller. We also observe from Figure 5-Figure 7 that for the SINR-MI there is no coupling in range and Doppler frequency, as is usually noticed in conventional chirp waveforms. It's also worth noting that for SINR-Matched Illumination waveforms (Table-1) the side-lobes appear, this is consistent with radar uncertainty principle since the main lobe gets compressed, hence spurious peaks appear to maintain the constant volume property.

VII. Conclusion

We conclude that SINR-MI waveforms provide much improved range and Doppler resolutions compared to conventional radar waveforms for the detection of range spread targets.

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